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A Jackson-Type Theorem for Averages of Splines Bounding a Class of Differentiable Functions

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This note gives a variation of a Jackson-type theorem by Andreev, Popov, and Sendov [1] for best one-sided approximations by splines. The main theorem states how well a function can be approximated by the average of upper and lower bounding splines of a given order with respect to the $L^p = L^p[0, 1]$ norm of a modified modulus of continuity.

In general, we consider functions whose kth derivatives are bounded on $[0, 1]$. Let τ denote the knot sequence $\{0 = x_0 < x_1 < \cdots < x_n = 1\}$. The ith normalized basic spline of order k with the knot sequence τ is denoted by $M_{i,k,\tau}$ and is defined by

$$M_i(x) = M_{i,k,\tau}(x) = \sum_{j=i}^{i+k} \left[\prod_{\substack{m=i \\ m \neq j}}^{i+k} \frac{1}{(x_j - x_m)} \right] k(x_j - x)_+^{k-1}$$

where

$$(x-t)_+^{k-1} = (x-t)^{k-1}, \qquad x \geqslant t,$$
$$= 0, \qquad x < t;$$

$i = 0, 1, ..., n-k$. It is well known [2] that $M_i(x) > 0$ for $x \in (x_i, x_{i+k})$, $M_i(x) = 0$ for $x \notin (x_i, x_{i+k})$, and $\int_{-\infty}^{\infty} M_i(x)\, dx = 1$. Let $\mathscr{S}_{k,\tau}$ be the linear space of all kth degree splines on the interval $[0, 1]$ with knot sequence τ. That is, s belongs to $\mathscr{S}_{k,\tau}$ if $s \in C^{k-1}[0, 1]$ and its restriction on $[x_i, x_{i+1}]$ is an algebraic polynomial of degree k for $i = 0, 1, ..., n-1$. Clearly $M_i(x) \in \mathscr{S}_{k-1,\tau}$. The problem of best one-sided L^p $(1 \leqslant p \leqslant \infty)$ approximation by splines in $\mathscr{S}_{k,\tau}$ for functions f defined on $[0, 1]$ is the study of

$$E_{k,\tau}(f)_{L^p} = \inf_{s,l} \left[\int_0^1 (s(x) - l(x))^p\, dx \right]^{1/p} \qquad (p \geqslant 1)$$

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and

$$E_{k,\tau}(f)_{L^\infty} = \inf_{s,l} \sup_x (s(x) - l(x)), \qquad x \in [0, 1]$$

where $l, s \in \mathscr{S}_{k,\tau}$ satisfy $l(x) \leqslant f(x) \leqslant s(x)$ for all $x \in [0, 1]$. See [1].

The specific aim of this note is to consider a related L^p approximation problem $F_{k,\tau}(f)_{L^p}$ (for $1 \leqslant p \leqslant \infty$) with respect to the average of bounding splines and derive a Jackson-type theorem relating the errors $F_{k,\tau}$, $E_{k,\tau}$ for L^p $(1 \leqslant p \leqslant \infty)$ and a modified modulus of continuity defined by

$$w(f; x; \delta) = \sup \left\{ |f(t_1) - f(t_2)| : |t_1 - x| \leqslant \frac{\delta}{2}, |t_2 - x| \leqslant \frac{\delta}{2} \right\}, \; t_1, t_2$$

belong to the domain of f.

More specifically, we consider the following:

DEFINITION 1.

$$F_{k,\tau}(f)_{L^p} = \inf_{l,s} \left(\int_0^1 \left| f(x) - \frac{l(x) + s(x)}{2} \right|^p dx \right)^{1/p}$$

$$F_{k,\tau}(f)_{L^\infty} = \inf_{l,s} \sup_x \left| f(x) - \frac{l(x) + s(x)}{2} \right|$$

where $l, s \in \mathscr{S}_{k,\tau}$ and $l(x) \leqslant f(x) \leqslant s(x)$ for all x in $[0, 1]$.

We need the following:

LEMMA 1. *Let f have an integrable bounded derivative f' on* $[0, 1]$, *and* $\tau = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ *with* $\Delta_n = \max\{|x_{i+1} - x_i|, 0 \leqslant i \leqslant n-1\}$. *Let* $1 \leqslant p \leqslant \infty$. *Then for any integer* $k \geqslant 1$,

(1) $F_{k,\tau}(f)_{L^p} \leqslant \frac{1}{2} E_{k,\tau}(f)_{L^p}$,

(2) $F_{k,\tau}(f)_{L^p} \leqslant (k+1)\, \Delta_n F_{k-1,\tau}(f')_{L^p}$, *and*

(3) $F_{k,\tau}(f)_{L^p} \leqslant ((k+1)/2)\, \Delta_n E_{k-1,\tau}(f')_{L^p}$.

Proof. Part (1) follows immediately from the inequality $|f(x) - (l(x) + s(x))/2| \leqslant (s(x) - l(x))/2$ for $l, s \in \mathscr{S}_{k,\tau}$ such that $l(x) \leqslant f(x) \leqslant s(x)$ for all x in $[0, 1]$.

The proof of part (2) is similar to the constructive proof in [1, Lemma 10, p. 893]. Since $\mathscr{S}_{k,\tau}$ contains constants, we can assume without loss of generality that $f(0) = 0$. Let $\varepsilon > 0$ and find $l^*, s^* \in \mathscr{S}_{k-1,\tau}$ such that $l^*(x) \leqslant f'(x) \leqslant s^*(x)$ for x in $[0, 1]$ and

$$\left\| f'(x) - \frac{l^*(x) + s^*(x)}{2} \right\|_{L^p} \leqslant F_{k-1,\tau}(f')_{L^p} + \varepsilon.$$

Set

$$\alpha_i = \int_{x_i}^{x_{i+1}} (f'(x) - s^*(x))\, dx \leqslant 0$$

$$\beta_i = \int_{x_i}^{x_{i+1}} (f'(x) - l^*(x))\, dx \geqslant 0, \qquad i = 0, 1, \ldots, n-1.$$

For $x \in [0, 1]$, define

$$s(x) = \int_0^x s^*(t)\, dt + \sum_{i=0}^{n-k-1} \alpha_i \int_0^x M_{i+1}(t)\, dt$$

$$l(x) = \int_0^x l^*(t)\, dt + \sum_{i=0}^{n-k-1} \beta_i \int_0^x M_{i+1}(t)\, dt.$$

Clearly l and $s \in \mathscr{S}_{k,\tau}$. Now we show that $l(x) \leqslant f(x) \leqslant s(x)$ for x in $[0, 1]$. Let $x \in (x_{i^*}, x_{i^*+1})$. Then

$$\begin{aligned} f(x) - l(x) &= \int_0^x f'(t)\, dt - l(x) \\ &= \int_0^x [f'(t) - l^*(t)]\, dt - \sum_{i=0}^{n-k-1} \beta_i \int_0^x M_{i+1}(t)\, dt \\ &= \sum_{i=0}^{i^*-1} \beta_i + \int_{x_i^*}^x [f'(t) - l^*(t)]\, dt \\ &\quad - \sum_{i=0}^{i^*-k-1} \beta_i - \sum_{i=i^*-k}^{i^*-1} \beta_i \int_0^x M_{i+1}(t)\, dt \\ &= \int_{x_i^*}^x [f'(t) - l^*(t)]\, dt + \sum_{i=i^*-k}^{i^*-1} \beta_i \left[1 - \int_0^x M_{i+1}(t)\, dt\right] \geqslant 0 \end{aligned}$$

as the integrands and the β_i are greater than or equal to zero and $\int_0^x M_{i+1}(t)\, dt \leqslant 1$.

Next

$$\begin{aligned} s(x) - f(x) &= s(x) - \int_0^x f'(t)\, dt \\ &= \int_0^x (s^*(t) - f'(t))\, dt + \sum_{i=0}^{n-k-1} \alpha_i \int_0^x M_{i+1}(t)\, dt \\ &= \sum_{i=0}^{i^*-1} (-\alpha_i) + \int_{x_i^*}^x (s^*(t) - f'(t))\, dt \end{aligned}$$

$$+\sum_{i=0}^{i^*-k-1}\alpha_i+\sum_{i=i^*-k}^{i^*-1}\alpha_i\int_0^x M_{i+1}(t)\,dt$$

$$=\int_{x_i^*}^x(s^*(t)-f'(t))\,dt-\sum_{i=i^*-k}^{i^*-1}\alpha_i\left(1-\int_0^x M_{i+1}(t)\,dt\right)\geqslant 0$$

as $\alpha_i\leqslant 0$ by definition and $\int_0^x M_{i+1}(t)\,dt\leqslant 1$.

So for $p\geqslant 1$, we write

$$\left\|f(x)-\frac{l(x)+s(x)}{2}\right\|_{L^p}=\left(\int_0^1\left|f(x)-\frac{l(x)+s(x)}{2}\right|^p dx\right)^{1/p}$$

$$=\left(\int_0^1\left|\int_{x_i^*}^x\left[f'(t)-\frac{l^*(t)+s^*(t)}{2}\right]dt\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{i^*-k}^{i^*-1}(\alpha_i+\beta_i)\left[1-\int_0^x M_{i+1}(t)\,dt\right]\right|^p dx\right)^{1/p}.$$

By the triangular inequality, basic integral properties, and $(1-\int_0^x M_{i+1}(t)\,dt)\leqslant 1$ for all i, we have

$$\leqslant\left(\int_0^1\left[\left|\int_{x_{i^*}}^x\left[f'(t)-\frac{l^*(t)+s^*(t)}{2}\right]dt\right|\right.\right.$$

$$\left.\left.+\frac{1}{2}\left|\sum_{i^*-k}^{i^*-1}(\alpha_i+\beta_i)\left[1-\int_0^x M_{i+1}(t)\,dt\right]\right|\right]^p dx\right)^{1/p}$$

$$\leqslant\left(\int_0^1\left[\int_{x_{i^*}}^x\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{i^*-k}^{i^*-1}|\alpha_i+\beta_i|\left|1-\int_0^x M_{i+1}(t)\,dt\right|\right]^p dx\right)^{1/p}$$

$$\leqslant\left(\int_0^1\left[\int_{x_{i^*}}^x\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt+\frac{1}{2}\sum_{i^*-k}^{i^*-1}|\alpha_i+\beta_i|\right]^p dx\right)^{1/p}$$

$$=\left(\int_0^1\left[\int_{x_{i^*}}^x\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\right.\right.$$

$$\left.\left.+\frac{1}{2}\sum_{i^*-k}^{i^*-1}\left|\int_{x_i}^{x_{i+1}}\left(f'(x)-\frac{l^*(t)+s^*(t)}{2}\right)dt\right|\right]^p dx\right)^{1/p}$$

$$\leqslant\left(\int_0^1\left[\int_{x_{i^*}}^x\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\right.\right.$$

$$+\frac{1}{2}\sum_{i^*-k}^{i^*-1}\int_{x_i}^{x_{i+1}}\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\Bigg]^p dx\Bigg)^{1/p}$$

$$=\left(\int_0^1\left[\int_{x_{i-k}^*}^{x}\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\right]^p dx\right)^{1/p}$$

$$\leqslant\left(\int_0^1\left[\int_{x-(k+1)\Delta_n}^{x}\left|f'(t)-\frac{l^*(t)+s^*(t)}{2}\right|dt\right]^p dx\right)^{1/p}$$

$$=\left(\int_0^1\left[\int_0^{(k+1)\Delta_n}\left|f'(x-(k+1)\Delta_n+t)-\frac{l^*(x-(k+1)\Delta_n+t)+s^*(x-(k+1)\Delta_n+t)}{2}\right|dt\right]^p dx\right)^{1/p}$$

$$\leqslant\int_0^{(k+1)\Delta_n}\left(\int_0^1\left|f'(x-(k+1)\Delta_n+t)-\frac{l^*(x-(k+1)\Delta_n+t)+s^*(x-(k+1)\Delta_n+t)}{2}\right|^p dx\right)^{1/p}dt$$

by the Minkowski integral inequality [3, p. 271],

$$\leqslant\int_0^{(k+1)\Delta_n}[F_{k-1,\tau}(f')_{L^p}+\varepsilon]\,dt,$$

by assumption,

$$=(k+1)\,\Delta_n[F_{k-1,\tau}(f')_{L^p}+\varepsilon].$$

This proves part (2) for $p\geqslant 1$.

For $p=\infty$ we have by a similar argument,

$$\left\|f(x)-\frac{l(x)+s(x)}{2}\right\|_{L^\infty}=\sup_x\left|f(x)-\frac{l(x)+s(x)}{2}\right|$$

$$\leqslant\sup_x\int_0^{(k+1)\Delta_n}\left|f'(x-(k+1)\Delta_n+t)-\frac{l^*(x-(k+1)\Delta_n+t)+s^*(x-(k+1)\Delta_n+t)}{2}\right|dt$$

$$\leqslant\int_0^{(k+1)\Delta_n}\sup_x\left|f'(x-(k+1)\Delta_n+t)-\frac{l^*(x-(k+1)\Delta_n+t)+s^*(x-(k+1)\Delta_n+t)}{2}\right|dt$$

$$\leqslant \int_0^{(k+1)\Delta_n} [F_{k-1,\tau}(f')_{L^\infty} + \varepsilon]\, dt$$

$$= (k+1)\, \Delta_n F_{k-1,\tau}(f')_{L^\infty} + \varepsilon \qquad \text{for all } \varepsilon > 0.$$

And so part (2) is proved.

Part (3) follows immediately by applying parts (2) and (1) successively. We can now prove the following:

THEOREM 1. *If f is bounded on $[0, 1]$, then for $1 \leqslant p \leqslant \infty$*

$$F_{0,\tau}(f)_{L^p} \leqslant \|w(f; x; \Delta_n)\|_{L^p}.$$

Proof. Let $\tau = \{0 = x_0 < \cdots < x_n = 1\}$ and $\Delta_n = \max_i |x_i - x_{i-1}|$, $i = 1,..., n$. Set

$$s_\tau(x) = \sup_t f(t), \qquad x \in [x_{i-1}, x_i),\ t \in [x_{i-1}, x_i]$$

$$s_\tau(1) = \lim_{x \to 1} s_\tau(x)$$

and

$$l_\tau(x) = \inf_\tau f(t), \qquad x \in [x_{i-1}, x_i),\ t \in [x_{i-1}, x_i]$$

$$l_\tau(1) = \lim_{x \to 1} l_\tau(x).$$

Also let

$$S(f, x; \delta) = \sup f(t) \qquad \text{where} \quad |t - x| \leqslant \delta/2$$

$$I(f, x; \sigma) = \inf f(t) \qquad \text{where} \quad |t - x| \leqslant \delta/2.$$

It follows immediately that

$$I(f, x; 2\Delta_n) \leqslant l_\tau(x) \leqslant f(x) \leqslant s_\tau(x) \leqslant S(f, x; 2\Delta_n).$$

We observe that $s_\tau, l_\tau \in S_{0,\tau}$, and

$$w(f, x, \delta) = S(f, x; \delta) - I(f, x; \delta).$$

Also we will need the property ([1, Lemma 5, p. 890]) that for any Riemann-integrable function f and integer k

$$\|w(f; x; k\delta)\|_{L^p} \leqslant k\, \|w(f; x; \delta)\|_{L^p}.$$

So finally

$$F_{0,\tau}(f)_{L^p} \leqslant \left(\int_0^1 \left| f(t) - \frac{s_\tau(t) + l_\tau(t)}{2} \right|^p dt\right)^{1/p}$$
$$\leqslant \tfrac{1}{2}\left(\int_0^1 [s_\tau(t) - l_\tau(t)]^p \, dt\right)^{1/p}$$
$$\leqslant \tfrac{1}{2}\left(\int_0^1 [S(f; x; 2\Delta_n) - I(f; x; 2\Delta_n)]^p \, dt\right)^{1/p}$$
$$= \tfrac{1}{2} \|w(f; x; 2\Delta_n)\|_{L^p} \leqslant \|w(f; x; \Delta_n)\|_{L^p}.$$

Our next results follow from our above work and results cited in [1].

THEOREM. *Let $f \in C^{k-1}[0, 1]$ and have an integrable bounded kth derivative on $[0, 1]$. Then for $1 \leqslant p \leqslant \infty$*

$$F_{k,\tau}(f)_{L^p} \leqslant (k+1)! \, \Delta_n^k \, \|w(f^{(k)}; x; \Delta_n)\|_{L^p}.$$

Proof. Using Lemma 1, part (2), k times and then Theorem 1 yields

$$F_{k,\tau}(f)_{L^p} \leqslant (k+1) \, \Delta_n F_{k-1,\tau}(f')_{L^p} \leqslant \cdots \leqslant (k+1)! \, \Delta_n^k F_{0,\tau}(f^{(k)})_{L^p}$$
$$\leqslant (k+1)! \, \Delta_n^k \, \|w(f^{(k)}; x; \Delta_n)\|_{L^p}.$$

COROLLARY 1. *Let $f \in C^{k-1}[0, 1]$ and have an integrable bounded kth derivative $f^{(k)}$; then*

(1) *$F_{k,\tau}(f)_{L^\infty} \leqslant (k+1)! \, \Delta_n^k \mu(f^{(k)}; \Delta_n)$ where μ is the modulus of continuity, $\mu(f; \delta) = \sup |f(x) - f(y)|$ and $|x - y| \leqslant \delta$.*

(2) *Also if f is of bounded variation on $[0, 1]$, then $F_{k,\tau}(f)_L \leqslant (k+1)! \, \Delta_n^{k+1} V_0^1(f^{(k)})$, where V_0^1 denotes the variation of the function on $[0, 1]$.*

Proof. First we observe that the best one-sided uniform approximation from a linear approximating family which includes constants is obtained by translating the best uniform approximation (either up or down). So the error thus obtained will be exactly twice the unconstrained error. Therefore the average $(l(x) + s(x))/2$, where $l(x)$, $s(x)$ are specified above, is the best uniform approximation in this case. The proof follows immediately from part 2 of Lemma 1; $\|w(f; x; \delta)\|_{L^\infty} = \mu(f; \delta)$ and [1, Corollary 1, p. 894] where the analogous inequalities for $E_{k,\tau}$ are

$$E_{k,\tau}(f)_{L^\infty} \leqslant 2(k+1)! \, \Delta_n^k \mu(f^{(k)}; \Delta_n)$$

and

$$E_{k,\tau}(f)_{L^1} \leqslant 2(k+1)! \, \Delta_n^{k+1} V_0^1 f^{(k)}.$$

Part (2) of Corollary 1 is analogous to a Freund–Popov theorem [4] for $E_{k,\tau}$.

The next corollary is an analogue of a Babenko–Ligun theorem [5] for $E_{k,\tau}$ where the interval $[0, 1]$ is replaced by $[0, 2\pi]$.

COROLLARY 2. *Let* $f \in C^k[0, 2\pi]$, $\|f^{(k+1)}\|_{L^p} < \infty$, *and* $\sigma = \{j\,2\pi/n\}_{j=0}^{n}$; *then*

$$F_{k,\sigma}(f)_{L^p} \leqslant C(k)\ \|f^{(k+1)}\|_{L^p}\, n^{-k-1}.$$

Proof. Again the proof follows immediately from Lemma 1, part (1), and

$$E_{k,\sigma}(f)_{L^p} \leqslant C(k)\ \|f^{(k+1)}\|_{L^p}\, n^{-k-1}.$$

In conclusion we remark that further generalizations of Theorem 1 can be obtained by weakening conditions on modulus of continuity, or by extending it to $\mathbb{R}^n$ using techniques in the preprint by Popov and Szabados [6].

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